Lighthouse Problem (Written component)

Here we will consider the lighthouse problem, which was discussed in class as well as in the text Sivia and Skilling. Consider that a lighthouse is located at some distance along the coastline and some distance offshore. The lighthouse rotates at a constant rate and can fire a pencil thin beam of light at random in any direction (uniformly distributed).

The lighthouse flashes hit the coast 64 times resulting in detections at the following locations:

\[ D = [4.73, 0.45, -1.73, 1.09, 2.19, 0.12, 1.31, 1.00, 1.32, 1.07, 0.86, -0.49, -2.59, 1.73, 2.11, 1.61, \ldots, 4.98, 1.71, 2.23, -57.20, 0.96, 1.25, -1.56, 2.45, 1.19, 2.17, -10.66, 1.91, -4.16, 1.92, 0.10, 1.98, \ldots, 2.51, 5.55, -0.47, 1.91, 0.95, -0.78, -0.84, 1.72, -0.01, 1.48, 2.70, 1.21, 4.41, -4.79, 1.33, 0.81, \ldots, 0.20, 1.58, 1.29, 16.19, 2.75, -2.38, -1.79, 6.50, -18.53, 0.72, 0.94, 3.64, 1.94, -0.11, 1.57, 0.57]; \]

1. The orientation of the lighthouse is described by the angle $\theta$ such that $\theta = 0$ is aimed directly at (perpendicular to) the coastline. Similarly, directions $\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$ are oriented parallel to the coastline.

   a. Consider that the beam can hit a point along the shore ($-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$) with uniform probability. What is the normalized probability density for the angle, $P(\theta | I)$, in this case?

   b. Adopt a model in which the lighthouse is located at position $(x_0, y_0)$. Perform a change of variables to find the probability density of the beam hitting at position $x$ along the shore. This is the likelihood $P(x | x_0, y_0, I)$ of detecting the beam at position $x$ given the model parameters $(x_0, y_0)$ describing the lighthouse location.

   c. Write the joint likelihood for the $N = 64$ light flash positions $\{x_1, x_2, \ldots, x_N\}$.

   d. Write Bayes’ Theorem for the posterior probability $P(x_0, y_0 | \{x_1, x_2, \ldots, x_N\}, I)$ writing the evidence simply as the constant $Z$.

2. a. Write the logarithm of the posterior probability, $\log P$, found in 1d.

   b. Find the first derivative of $\log P$ with respect to both parameter values, $x_0$ and $y_0$.

   c. Find the second derivatives: $\frac{\partial^2}{\partial x_0^2} \log P$, $\frac{\partial^2}{\partial y_0^2} \log P$, $\frac{\partial^2}{\partial x_0 \partial y_0} \log P$.

3. Show that by setting the first derivatives equal to zero, that it is not possible to analytically solve for the most probable lighthouse position $(\hat{x}_0, \hat{y}_0)$.

4. We will now attempt to find a numerical solution while taking advantage of the analytical expressions for the second derivatives to obtain an analytical estimate of the uncertainty.

   a. Use the function fminsearch in Matlab to find the most probable position, $(\hat{x}_0, \hat{y}_0)$, of the lighthouse by finding the maximum of the log posterior $\log P$.

   b. Evaluate the second derivatives at the most probable position $(\hat{x}_0, \hat{y}_0)$, and use these to find the uncertainties in the position estimates writing the final solution as $\hat{x}_0 \pm \sigma_{x_0}$ and $\hat{y}_0 \pm \sigma_{y_0}$. 