SUMMARY

We will use two techniques to measure the radius of a set of identical circles illustrated on a sheet of paper. This experiment will be used to demonstrate how multiple measurements can be combined to compute estimates of physical quantities as well as the associated uncertainties in those estimates.

The first technique we will employ will be to either throw/drop a pen or pencil at/onto the sheet of paper. We can treat this as a scattering experiment where one can label each event as a hit or a miss. From the number of hits and misses we will estimate both the radius and the area of the circles as well as the uncertainties in those estimates.

The second technique will employ a measuring stick or ruler where one will measure the diameter of the circles directly. From this list of measurements, we will come up with an alternate estimate of both the radius and area of the circles along with uncertainties.

THEORY

Physics relies on evaluating the ability of competing theories to make predictions about measurements or data in general. Consider an example where Theory A predicts that a measurement of the radius of a circle is 5cm and Theory B predicts that it should be 4cm. You carry out an experiment and take a measurement to find that you obtained the circle diameter to be 4.4cm. Clearly this measurement does not agree with either theory. But does it rule out one theory or both?

Measurements aren’t perfect. Not all situations are identical. Not all equipment is accurately calibrated. That is, errors can come from many sources: **systematic error (bias)**, **measurement precision**, or **statistical variation**. Systematic error or bias can arise from not taking into account an important physical effect in the experiment, comparing the measurements to a theory that makes inappropriate assumptions (which is very similar to the first example), or working with poorly calibrated equipment. Measurement precision reflects one’s ability to make measurements and is quantified by a measurement uncertainty. Statistical variation occurs in cases where there are many trials and one’s inability to make sure that the trials are identical as assumed.

When reporting results, one must be mindful of the uncertainties in the resulting estimates and this must be reflected in the number of reported significant digits. A discussion of significant digits can be found in Cutnell and Johnson appendix A-1 or in Serway chapter 1.

We will rely on some specific formulae to compute estimates and uncertainties in a variety of situations. Many of these will be provided in handouts. All can be found in Bayesian Data Analysis: A Tutorial by Sivia (first edition) or Sivia and Skilling (second edition).
CHARACTERIZING CIRCLES VIA SCATTERING

EXPERIMENT INTRODUCTION
A rectangular region with area A on a sheet of paper has K circles drawn on it. Each circle has radius r. The circles do not overlap or intersect. This is all you know. The picture to the right is for illustrative purposes only.
Use the sheet supplied at the end of this writeup.

EXPERIMENT A
PROCEDURE
Print out a high quality copy of the sheet with circles on the last page of this document.

We will basically estimate the area of the circles by throwing/dropping a pen or pencil at/onto the sheet of paper and counting how many times the impact falls within the circle. Another method would be to drop very small pebbles onto the sheet of paper and note how many times a pebble falls within one of the circles. I will refer to pebbles in the analysis below.

What to do with the finite boundary is up to you. Be consistent and take this in account in your analysis and technical report. Do the experiment carefully and note the difficulties this method presents, and how you chose to overcome them.

ANALYSIS
We are going to employ Bayes Theorem to derive the probability that the circles have area S given that that M pebbles fall within a circle out of N pebbles total. That is we will compute $P(S | A, M, N, K, I)$ where I is simply a symbol which represents the prior information you possess. Bayes Theorem is then given by the proportionality

$$P(S | A, M, N, K, I) \propto P(S | I) P(M | A, N, S, K, I)$$

which is a product of the probability that the circles have an area S and the likelihood (probability) of obtaining M hits out of N throws given that the sheet has area A and that the circles have area S. We will assign a constant probability for the prior probability of the area of the circle over some given range, and this simplifies the proportionality above to

$$P(S | A, M, N, K, I) \propto P(M | A, N, S, K, I)$$

We will figure out the formula for $P(M | A, N, S, K, I)$ in stages by solving the following problems:

A boy drops small pebbles (stones) onto the sheet of paper. Sometimes a stone lands in a circle. Let A be the area of the sheet of paper (don’t plug in number for A).
Let C1 represent the statement that the pebble lands in circle number 1.
Let N=1 represent the statement that the boy has dropped one pebble.
What is the likelihood $P(C1 | A, N = 1, S, I)$ that the pebble lands on circle number 1?
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Given that the result \( P(C1 \mid A, N = 1, S, I) \) has the same form for any single circle \( C# \), find the likelihood \( P(C1 \lor C2 \mid A, N = 1, S, I) \) that the pebble could have landed in either circle 1 (\( C1 \)) or circle 2 (\( C2 \))?

From now on, we will write this as \( P(C \mid A, N = 1, S, K = 2, I) \) where \( C \) means that it landed in one of the \( K=2 \) circles.

Let \( C = C1 \lor C2 \lor C3 \lor \ldots \lor CK \) represent the statement that the pebble lands in any one of the \( K \) circles. What is the likelihood \( P(C \mid A, N = 1, S, K, I) = P(C1 \lor C2 \lor C3 \lor \ldots \lor CK \mid A, N = 1, S, K, I) \)?

Let \( \sim C \) represent the statement that the pebble did not land in any circle. What is the likelihood \( P(\sim C \mid A, N = 1, S, K, I) \)?

If two pebbles are dropped, the likelihood that both pebbles fall inside a circle is
\[
P(C \mid A, N = 1, S, K, I) \times P(C \mid A, N = 1, S, K, I).
\]
If two pebbles are dropped, the likelihood that only one pebble falls in a circle is
\[
P(C \mid A, N = 1, S, K, I) \times P(\sim C \mid A, N = 1, S, K, I) + P(C \mid A, N = 1, S, K, I) \times P(\sim C \mid A, N = 1, S, K, I).
\]
Can you explain this?

For \( N \) pebbles, show that the likelihood that \( M \) of them will fall inside a circle is
\[
P(M \mid A, N, S, K, I) = \frac{N!}{M! (N-M)!} P(C \mid A, N, S, K, I)^M \times P(\sim C \mid A, N, S, K, I)^{N-M}
\]
where \( N \) represents the fact that \( N \) pebbles were dropped and \( M \) of those fell inside a circle.

Obtain the posterior probability (up to a normalization factor) by writing Bayes Theorem as a proportionality of your prior times your likelihood \( P(S \mid A, M, N, K, I) \propto P(M \mid A, N, S, K, I) \).

The goal now is to find the most probable circle area \( S \). It is far easier to look at the logarithm of the probability, which has a maximum at the same value of \( S \).

Write the logarithm of \( P(S \mid A, M, N, K, I) \).

Take the derivative of \( \log P(S \mid A, M, N, K, I) \) with respect to \( S \) and set it to zero. Solving for \( S \) gives you the most probable value of the area of the circle \( S \).

Take the counts \( M \) you obtained in your \( N \) throws and find the most probable value of \( S \) and label it \( \hat{S} \).
This is your best estimate for the area of the circle \( S \).

Plot \( \log P(S \mid A, M, N, K, I) \) versus \( S \) to verify that your solution \( \hat{S} \) is a peak.
CHARACTERIZING CIRCLES VIA SCATTERING

Now for the uncertainty in your estimate. Take the second derivative of \( \log P(S \mid A, M, N, K, I) \) with respect to \( S \) and evaluate it at \( \hat{S} \). That is, compute

\[
H = \frac{\partial^2}{\partial S^2} \log P(S \mid A, M, N, K, I) \bigg|_{S=\hat{S}}
\]

The uncertainty in \( S \) is then given by

\[
\sigma = \sqrt{-\frac{1}{H}}.
\]

As a rule of thumb include two significant digits of accuracy in your reported value of \( \sigma \), and as many significant digits in \( \hat{S} \) that \( \sigma \) warrants. Note that this may not quite agree with what is recommended in Cutnell and Johnson appendix A-1 or in Serway chapter 1, which may be more conservative.

Your final estimate is then expressed by \( \hat{S} \pm \sigma \).

Note that \( \sigma \) represents the precision in your estimate... not the accuracy of your results!

Estimate the radius of a circle as well as the uncertainty. Ideally, you should go through all of the calculations again writing \( S = \pi R^2 \) and taking derivatives with respect to \( R \) rather than \( S \).

Discuss.

EXPERIMENT B

PROCEDURE

Using the same high quality copy of the sheet with circles, take a meter stick and measure the diameter of the circle to the best of your ability. You will take multiple measurements (at least 10) and use these to estimate the area of the circles. Note that there are several potential situations that could lead to systematic bias. Do your best to avoid them.

Using these results, estimate both the radius and area of the circles along with the uncertainty in the estimates.

Given these results, how many pebbles would you have to drop in Experiment A to get the level of uncertainty obtained in Experiment B?